THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and thirty-third meeting of the Society was held in New York City on Saturday, April 27, 1907. The April meeting is usually well attended, and ranks next in importance to the annual and the summer meetings. The present occasion was especially marked by Professor W. F. Osgood's presidential address, the subject of which was "The calculus in our colleges and technical schools." Under this attracting influence the attendance exceeded all previous records for the April meetings, reaching about seventy persons including the following sixty-one members of the Society:

Miss Grace Andrews, Professor G. A. Bliss, Professor Joseph Bowden, Professor E. W. Brown, Dr. W. H. Bussey, Dr. J. E. Clarke, Miss Emily Coddington, Professor F. N. Cole, Miss E. B. Cowley, Professor T. W. Edmondson, Professor L. P. Eisenhart, Professor F. C. Ferry, Professor T. S. Fiske, Dr. F. L. Griffin, Mr. G. W. Hartwell, Dr. G. W. Hill, Dr. A. M. Hiltebeitel, Dr. A. A. Himowich, Professor E. V. Huntington, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. G. H. Ling, Mr. Joseph Lipke, Mr. L. L. Locke, Dr. W. R. Longley, Professor E. O. Lovett, Mr. E. B. Lytle, Professor T. E. McKinney, Professor Max Mason, Dr. R. L. Moore, Professor Richard Morris, Professor W. F. Osgood, Dr. F. M. Pedersen, Miss Amy Rayson, Mr. H. W. Reddick, Dr. R. G. D. Richardson, Miss S. F. Richardson, Miss I. M. Schottenfels, Mr. L. P. Siceloff, Professor C. S. Slichter, Dr. Clara E. Smith, Professor D. E. Smith, Mr. F. H. Smith, Professor P. F. Smith, Dr. R. P. Stephens, Dr. C. E. Stromquist, Dr. W. M. Strong, Professor H. D. Thompson, Mr. M. O. Tripp, Professor H. W. Tyler, Dr. A. L. Underhill, Mr. C. B. Upton, Professor J. M. Van Vleck, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams, Professor E. B. Wilson, Professor J. E. Wright, Professor J. W. Young.

The President of the Society, Professor H. S. White, occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. Alfred Ackermann-Teubner, Leipzig, Germany; Dr. J. W. Bradshaw, University of Michigan; Professor H. E. Cobb, Lewis

Institute, Chicago, Ill.; Mr. S. A. Corey, Hiteman, Ia.; Professor Floyd Field, Georgia School of Technology; Mr. G. W. Hartwell, Columbia University; Chancellor C. C. Jones, University of New Brunswick; Mr. Joseph Lipke, Columbia University; Professor Francis Regis, Christian Brothers College, St. Louis, Mo.; Mr. H. P. Stellwagen, Yeatman High School, St. Louis, Mo. Seven applications for admission to the Society were received. The total membership on May 1 was 560.

Following the plan announced in the Secretary's report of the February meeting, abstracts of the papers, as far as available, had been printed and issued with the programme in advance of the meeting. In this way it is hoped to secure a more intelligent interest in the papers and to promote criticism and discussion..

The date of the summer meeting, to be held at Cornell University, was fixed for Thursday–Friday, September 5–6.

The following papers were read at the April meeting:

(1) Professor G. A. BLISS: "A new form of the simplest problem of the calculus of variations."

(2) Professor R. D. CARMICHAEL: "Multiply perfect even numbers of five different primes" (preliminary communication).

(3) Professor L. P. EISENHART: "Transformations of surfaces whose lines of curvature are represented on the sphere by an isothermal system."

(4) Dr. F. L. GRIFFIN: "The variation of the apsidal angle in certain families of central orbits."

(5) Dr. F. L. GRIFFIN: "The solutions of central force problems as functions of the constant of areas."

(6) Dr. F. L. GRIFFIN: "Note on a simple example of a central orbit with more than two apsidal distances."

(7) Professor G. A. MILLER: "Note on the commutator of two operators."

(8) Professor J. E. WRIGHT: "Arrangement of ovals of a plane sextic curve."

(9) Professor W. F. OSGOOD, Presidential address: "The calculus in our colleges and technical schools."

(10) Miss I. M. SCHOTTENFELS: "Group matrices."

(11) Dr. C. E. STROMQUIST: "An inverse problem of the calculus of variations."

(12) Dr. R. G. D. RICHARDSON: "On the integration of a series term by term."

(13) Dr. A. L. UNDERHILL: "Invariants of the function

$F(x, y, x', y')$ under point and parameter transformations connected with the calculus of variations."

(14) Professor EDWARD KASNER: "The motion of particles under conservative forces."

(15) Professor EDWARD KASNER: "Isogonal and dynamical trajectories."

(16) Professor P. L. SAUREL: "On the distance from a point to a surface."

(17) Professor T. E. MCKINNEY: "On concyclic quantics."

(18) Professor T. E. MCKINNEY: "On continued fractions representing quadratic irrationalities."

(19) Professor G. A. MILLER: "Groups generated by n operators each of which is the product of the $n-1$ remaining ones."

In the absence of the authors the papers of Professor Carmichael, Professor Miller, Professor Saurel, and Professor McKinney were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The simplest problem of the calculus of variations in the plane has been studied in two different forms. In the earlier one the curves considered are represented by equations $y = y(x)$, where $y(x)$ is a single-valued function, and the integral to be minimized is

$$I = \int f(x, y, y')\, dx. \tag{1}$$

In the later form, which seems first to have been studied in detail by Weierstrass, the curves are taken in parametric representation $x = \phi(t)$, $y = \psi(t)$, and the integral in the form

$$I = \int F(x, y, x', y')\, dt, \tag{2}$$

where the function F is positively homogeneous in the derivatives x' and y'. Both methods are open to objections, the former because it is inconvenient to apply to curves for which the function $y(x)$ is not single-valued or for which the slope becomes infinite, and the latter on account of the homogeneity condition $F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y')$, $(\kappa > 0)$, which with its consequences must be kept in mind and frequently applied.

In a paper entitled "A generalization of the notion of angle" (*Transactions of the American Mathematical Society*, volume 7

(1906), page 184), Professor Bliss has introduced a third form of the problem in which the integral involves only invariants of the curve under change of parameter representation. The integral there given is

$$I = \int f(x, y, \tau)ds, \tag{3}$$

where s is the length of arc along the curve, and τ is the angle which the tangent to the curve makes with the x-axis, defined by the equations

$$\cos\tau = \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad \sin\tau = \frac{y'}{\sqrt{x'^2 + y'^2}}.$$

The integral (2) may be easily put into the form (3), for on account of the homogeneity of F,

$$F(x, y, x', y') = F(x, y, \cos\tau, \sin\tau)\sqrt{x'^2 + y'^2}.$$

But in (3) the function $f(x, y, \tau)$ need not be periodic as is the case in the right member of the last equation. Besides this generalization, which is important in applications to geometry such as Hamel's (see *Mathematische Annalen*, volume 57 (1903), page 231), the use of the integral in the form (3) avoids both of the objections made above to the other methods.

In the author's paper referred to above only the Euler equations and the equations defining transversality, which were necessary in finding the generalization of angle, were developed. In the present paper it is proposed to complete the theory of the problem in the form (3) by deriving the first and second variations and the functions which are commonly used in proving the usual necessary conditions in the calculus of variations (see Bolza, Lectures on the calculus of variations, pages 123, 133, 138; also Osgood, *Annals of Mathematics*, series 2, volume 2 (1901), pages 105 ff). The derivation of the conditions themselves can be made as is usual in the books on the subject.

2. By a rather difficult process, which he hopes to be able to simplify later, Professor Carmichael continues his investigations into the existence of multiply perfect numbers. In the present paper he takes up the case of seven numbers having only five different prime factors, finding that there are but four such numbers which are multiply perfect. One of these, $2^{13} \cdot 3 \cdot$

$11 \cdot 43 \cdot 127$, is of multiplicity 3; the others, $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, $2^7 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31$, $2^9 \cdot 3^3 \cdot 5 \cdot 11 \cdot 31$, are of multiplicity 4.

3. Ten years ago Thybaut* showed that minimal surfaces can be transformed into minimal surfaces in a manner analogous to the Bäcklund transformations of pseudospherical surfaces. A minimal surface and its Thybaut transform are the focal sheets of a W-congruence. The two minimal surfaces adjoint to these are the sheets of an envelope of spheres whose centers lie on a surface applicable to a paraboloid of revolution, by the theorem of Guichard. Darboux† and Bianchi‡ have considered the pairs of isothermic surfaces which are the envelope of a family of spheres and in conformal correspondence also, and the latter has given the name transformation of Darboux to a transformation of an isothermic surface which changes it into another such that the two form such a pair. Professor Eisenhart considers surfaces with isothermal representation of their lines of curvature and establishes a transformation of such a surface into another of the same kind. His transformation is a transformation of Darboux when the given surface is minimal and only in this case. Moreover, when the surface is minimal, this transformation is the same problem in analysis as the Thybaut transformation. By means of a theorem of Moutard it is shown that with each surface S whose lines of curvature have isothermal representation on the sphere, there are associated ∞^4 surfaces of the same kind, each of which forms with S the surface envelope of a family of spheres depending on two parameters. The determination of these new surfaces is the same problem apart from quadratures as the finding of the Thybaut transformations of the minimal surface with the same representation of its asymptotic lines as the lines of curvature of S. These transformations admit of a theorem of permutability similar to the theorems by this name shown by Bianchi to be true for Bäcklund and Darboux transformations. In consequence of this theorem, when all the transforms S_1 of a surface S are known, the transforms of the surfaces S_1 can be found without quadrature. Surfaces with plane lines of curvature are of the kind under discussion, as well as certain surfaces with spherical lines of curvature in one system. These

* *Annales de L'Ecole Normale*, ser. 3, vol. 14, pp. 43–98.

† *Annales de L'Ecole Normale*, ser. 3, vol. 16, pp. 465–508.

‡ *Annali di Matem.*, ser. 3, vol. 11, pp. 93–157.

two classes of surfaces and their transformations are studied at length.

4. In central orbits the apsidal angle usually depends upon the apsidal values of the radius vector.* Considering a family of orbits, whose pericentral distance is constant and whose apocentral distance varies, Dr. Griffin discusses the definite integral giving the apsidal angle. He shows that for any force f which is a function of the distance r, such that fr^2 and its derivative with respect to r are both decreasing functions of r, the apsidal angle is a decreasing function of the apocentral distance.

An application is made to the motion of a particle in the equatorial plane of an oblate spheroid, explaining the more rapid "advance of the line of apsides" in the more nearly circular orbits.†

5. While the important part played by the constant of areas h in determining the nature of a central orbit has received some consideration, little attention has been given to the form in which h appears in the solutions. Dr. Griffin's second paper deals with a family of orbits with fixed pericenter, described under a central force, such that the solutions are expressible as power series. The coefficients are shown to be polynomials in $1/h$, having some interesting properties.

The results offer a working basis for the discussion of certain convergence questions. In this connection, as a check, the solutions for the newtonian law are examined also by the method devised by Professor Moulton ‡ for ascertaining the true radii of convergence for related series.

6. It is well known § that orbits described under a central force which is a single-valued function of the distance are symmetric with respect to every apsidal line where the radius vector is a maximum or minimum. Consequently the apsidal angle is constant and the orbit has not more than two apsidal distances. These conclusions, however, may not be drawn when the force is not a single-valued function of the distance.

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* Routh, E. J., Dynamics of a particle. §§ 368–370.

† Moulton, F.R., Celestial mechanics, pp. 232–233.

‡ *Astronomical Journal*, vol. 23, 1903, pp. 93–102.

§ Routh, E. J. Dynamics of a particle, pp. 270–272.

In this note, Dr. Griffin considers the ellipse, and shows that with certain interior points as centers of force, the orbit has three distinct apsidal distances and two distinct apsidal angles. The force has the peculiarity that it does not permit a real orbit in all parts of the plane.

7. Professor Miller's first paper aims to exhibit the different ways in which the term commutator has been used in the literature of the group theory, and to point out some of the sources of these disagreements in the hope that these data may tend towards greater uniformity in the use of this term and also make its various meanings less confusing to the reader. It is observed that the definition given by Weber in his Lehrbuch der Algebra, volume 2 (1899), page 133, differs from the earlier definition of Dedekind and Frobenius, and that some writers use a definition according to which the commutator is not completely determined by means of its elements. The conclusion reached is that it would be desirable to define $t^{-1}sts^{-1}$ and $s^{-1}tst^{-1}$ as the commutators of s, t and of t, s respectively.

8. Professor Wright shows that if a plane sextic have the maximum number (11) of ovals, they cannot be all external to one another, nor can they lie ten inside the eleventh. His proof consists first in modifying the equation of a sextic supposed to have the given arrangement of ovals so as to produce a sextic with ten isolated points. A further process of modification is next introduced whereby finally a sextic is obtained with eleven isolated points. Such a sextic would be reducible and it is easily seen that it cannot exist. Other arrangements of the ovals are also considered.

9. Professor Osgood's presidential address appears in full in the present number of the BULLETIN.

10. In the BULLETIN for November, 1906, pages 81–84, Dr. Lehmer exhibits a most orderly arrangement for the permutations on n letters. Two important laws are developed for the group matrices of the symmetric, alternating, and metacyclic groups upon n letters and these laws necessitate the development of but one-fourth of the matrix.

In the present paper Miss Schottenfels presents a further study of these particular group matrices, and also diagrams showing the symmetric positions of the matrices of the most

important subgroups with respect to the original ones for $n = 2, 3, 4$, and 5.

11. The simplest problem in the calculus of variations is the following: * Given an integral

$$I = \int_{x_0}^{x_1} g(x, y, p)dx,$$

where $p = dy/dx$, to find a curve $y = y(x)$ which joins two given fixed points and renders this integral a minimum. As is well-known, the solutions $y = y(x)$ of the problem must satisfy Euler's † differential equation, which is a differential equation of the second order with a two-parameter family of solutions $y = y(x, \alpha, \beta)$ usually called extremals.

The so-called inverse problem arises when a two-parameter family of curves $y = y(x, \alpha, \beta)$ is given and we seek to determine the integral

$$I = \int_{x_0}^{x_1} g(x, y, p)dx$$

which shall have these curves for its extremals. This inverse problem has been discussed by Darboux,‡ and it turns out that there are in general an infinite number of solutions.

In Dr. Stromquist's paper a third problem in connection with the integral I is considered, viz., if at every point of the plane a relation between the direction p of an extremal and the direction P transverse to it be given in the form (1) $P = \theta(x, y, p)$, is it then possible to determine an integral I whose extremals and transversals satisfy the given relation? It is shown that an infinitude of functions g can always be determined which will give rise to the relation (1). Further it can be required that a one-parameter family of curves $y = y(x, \gamma)$ be extremals for the integral I. Then it follows that g is uniquely determined provided that in the field composed of extremals $y = y(x, \gamma)$ the value of I as a function of x be given along any curve which is not tangent at any point to a transversal of the field.

In the second part of the paper the above results are applied to some examples.

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* Bolza, Lectures on the calculus of variations, § 3.

† Bolza, loc. cit., p. 22.

‡ Darboux, Théorie des surfaces, vol. 3, p. 53. See also Bolza, loc. cit., p. 30.

12. Dr. Richardson's paper is in abstract as follows: Given a series $U(x) = u_1(x) + u_2(x) + \cdots$, what is a sufficient condition that integration term by term is permissible? Uniform convergence in a more or less modified form is the condition generally imposed, and the work of Osgood and Arzelà has this for a basis. For large classes of functions this condition is superfluous. Granted that the functions U, u_1, $u_2 \cdots$ are integrable, we may without any further condition write down three classes of series which are integrable term by term:

(a) The function u_1, $u_2 \cdots$ are all of the same sign ($u_i \geqq 0$, $i = 1, 2, 3, \cdots$; $u_i \leqq 0$).

(b) The series $\int |u_1| + \int |u_2| + \cdots$ converges.

(c) For all values of n the sum $\sum_1^n u_i$ is less in absolute value than some arbitrarily large but fixed number.

No restriction is put on the continuity of the functions U, u_1, $u_2 \cdots$, and in classes (a) and (b) these functions may take any values finite or infinite. An extension to functions of m variables is immediate.*

13. The object of Dr. Underhill's paper is the consideration of invariants, under point and parameter transformation, connected with the integrand function of the usual definite integral of the calculus of variations

$$I = \int_{t_0}^{t_1} F(x, y, x', y')dt.$$

The method developed is to start from a known invariant and from this, by a process not essentially different from the S-process of the calculus of variations, to derive new invariants.

Three invariants, denoted by S, K, V, are computed without restricting the arguments x, y, and their derivatives to those of an extremal. In case an extremal is used, a modified form of K is found and connected with it an invariantive normal form of the second variation of the integral I.

Finally S and K are interpreted for the problem of the geodesics on a surface. S is found to be the geodesic curvature; and K, when computed along an extremal, is the gaus-

* Osgood, "Non-uniform convergence and the integration of a series term by term" (*Amer. Journ. of Math.*, vol. 19.). Arzelà, Sulle serie di funzioni, Bologna, 1900.

sian curvature. If the value of K is computed along a curve which is not an extremal, it becomes equal to the gaussian curvature minus one half the square of the geodesic curvature.

Incidentally an expression for the Weierstrass function F_2 is found as well as a proof the Jacobi-Bonnet theorem: In the case of the problem of geodesics on a surface of negative curvature, the conjugate point is non-existent.

14. The theory of conservative forces worked out in Professor Kasner's first paper applies to any number of interacting particles, or to the equivalent problem of a single particle in a space of arbitrary dimensionality. The possible motions may be divided into families each characterized by a particular value of the constant of energy. To each value of this constant corresponds a definite system of trajectories, which is termed, following Painlevé, a natural system. The chief object of the paper is to find the geometric properties of such natural systems. The first result states that the circles of curvature constructed at a given point have another point in common. The point-to-point correspondence thus arising is not arbitrary; its characterization is given in the second property. If a system has both properties, it is necessarily natural. The results apply to the related theories of geodesics (Darboux, Théorie des surfaces, volume 2, chapter 8) and of contact transformations (Lie, Berührungstransformationen, page 102, and *Leipziger Berichte*, 1889).

15. Professor Kasner's second paper deals with certain analogies between the theory of natural systems of dynamical trajectories in the plane, and systems of isogonal trajectories. In both cases the circles of curvature at a given point form a pencil. For isogonals the theorem is due to Cesàro and Scheffers. The types differ however with respect to the nature of the induced point transformation. The author obtains complete characterizations for both types; connects the two theories by a certain transformation of curvature elements; establishes a simple reciprocity between dynamical systems; and finally proves that a natural system (and hence also a geodesic system) is completely determined by an arbitrary simple infinity of curves together with an arbitrary transversal curve.

16. Professor Saurel's paper appears in full in the present number of the BULLETIN.

17. In Professor McKinney's first paper are derived the sufficient conditions in terms of the invariants of the quantic that its roots, supposed distinct, be concyclic.

18. Professor McKinney's second paper discusses the properties of the continued fractions depending on a variable parameter λ, and representing quadratic irrationalities.

19. The principal theorems proved in Professor Miller's second paper are as follows: If the n different operators $s_1, s_2, \cdots, s_n$ are commutative and if each of them is the product of all the others, then they generate the direct product of a cyclic group whose order divides $2(n-2)$ and an abelian group of order 2^n and of type $(1, 1, 1, \cdots)$. Moreover, any such direct product may be generated by n operators which satisfy the given condition. If the order of one of these operators is divisible by 4, all of them are of the same order. In fact they always have a common square and the order of each divides $2(n-2)$. The necessary and sufficient condition that two different operators which have a common square are commutative is that one is the product of the other into an operator of order two.

If $s_1, s_2, \cdots, s_n$ are not supposed to be commutative but satisfy the condition $s_\alpha = s_{\alpha+1}s_{\alpha+2}\cdots s_n s_1 s_2 \cdots s_{\alpha-1}$ $(\alpha = 1, 2, \cdots, n)$ they may generate a large number of different types of groups when $n > 4$. In particular, every possible symmetric group whose degree exceeds 3 may be generated by five such operators. All of these operators must have a common square and $s_{n-1}s_{n-2}\cdots s_2 s_1 = s_n s_1^{2(n-2)}$. The present paper determines the possible groups for the case when $n = 4$, those for smaller values of n being known. It is proved that every group which may be obtained by extending the direct product of two cyclic groups by means of an operator of order two which transforms each operator of this direct product into its inverse may be generated by four operators of order 2, each of which is a product of the other three. A similar theorem is proved for the case when each of the operators s_1, s_2, s_3, s_4 is of order 4. It is observed that each of these four operators is always of even order and that the possible groups are somewhat similar to the dihedral type. In particular, all of them are solvable.

F. N. COLE,
Secretary.

THE MARCH MEETING OF THE CHICAGO SECTION.

THE twenty-first regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Saturday, March 30, 1907. The total attendance was fifty, including the following thirty-eight members of the Society:

Professor W. W. Beman, Mr. G. D. Birkhoff, Professor O. Bolza, Mr. W. E. Brooke, Professor C. E. Comstock, Professor D. R. Curtiss, Professor E. W. Davis, Professor L. E. Dickson, Dr. E. L. Dodd, Professor L. W. Dowling, Mr. E. P. R. Duval, Mr. E. B. Escott, Dr. Peter Field, Professor A. G. Hall, Professor C. N. Haskins, Professor E. R. Hedrick, Mr. L. Ingold, Professor O. D. Kellogg, Dr. H. G. Keppel, Professor Kurt Laves, Mr. N. J. Lennes, Mr. W. D. MacMillan, Professor H. Maschke, Professor G. A. Miller, Professor E. H. Moore, Dr. J. C. Morehead, Professor F. R. Moulton, Dr. L. I. Neikirk, Mr. F. W. Owens, Professor D. A. Rothrock, Professor W. J. Rusk, Mr. A. R. Schweitzer, Professor E. B. Skinner, Professor H. E. Slaught, Professor E. J. Townsend, Professor E. B. Van Vleck, Dr. A. E. Young, Professor J. W. A. Young.

The chairman of the section, Professor E. B. Van Vleck, called the meeting to order and presented Professor H. Maschke, Vice-President of the Society, who presided at both the morning and afternoon sessions. It was voted that hereafter, at the discretion of the programme committee, two days be given to both the April and December meetings of the Section. A full meeting of the enlarged committee on programme for the December, 1907, meeting was held and definite plans were formulated for a symposium on the teaching of engineering mathematics.

The following papers were read:

(1) Dr. L. I. NEIKIRK: "A geometric representation of the Galois field."

(2) Mr. A. R. SCHWEITZER: "On the Ausdehnungslehre of Grassmann."

(3) PROFESSOR E. W. DAVIS: "A geometric construction for the imaginary intersections of a conic and a line" (preliminary communication).

(4) PROFESSOR L. E. DICKSON: "Representations of the general symmetric group as linear groups in finite and infinite fields."

(5) PROFESSOR E. B. VAN VLECK: "Note on curves without tangents."

(6) Dr. A. E. YOUNG: "Asymptotic and isothermic surfaces."

(7) Mr. LOUIS INGOLD: "Vector theory in terms of symbolic differential parameters."

(8) Mr. E. B. ESCOTT: "New expressions for the number of classes of quadratic forms."

(9) PROFESSOR O. BOLZA: "Existence proof for a field of extremals tangent to a given curve."

(10) Dr. PETER FIELD: "Note on a certain plane quintic curve."

(11) PROFESSOR L. E. DICKSON: "The symmetric group on eight letters and the senary first hypoabelian group."

(12) Dr. C. H. SISAM: "On a property of parabolas and hyperbolas of mth degree."

(13) Mr. LOUIS INGOLD: "Vector differential geometry in n-space."

(14) Dr. E. L. DODD: "The iterated generalized limits of Cesàro and Borel."

(15) Mr. G. D. BIRKHOFF: "Boundary value problems of ordinary linear differential equations."

(16) PROFESSOR G. A. MILLER: "Groups defined by the orders of two operators and the order of their commutator."

(17) PROFESSOR O. BOLZA: "Concerning conjugate points in the case of discontinuous solutions in the calculus of variations."

(18) PROFESSOR KURT LAVES: "Jacobi's partial differential equations, II."

(19) PROFESSOR R. D. CARMICHAEL: "On certain quartic curves which may degenerate into an ellipse."

(20) PROFESSOR R. D. CARMICHAEL: "A table of the numbers corresponding to given values of Euler's function $\phi(m)$."

(21) PROFESSOR R. D. CARMICHAEL: "On constructing a cube having a given ratio to a given cube."

(22) Professor C. C. GROVE: "The complete Pappus hexagon."

(23) PROFESSOR E. R. HEDRICK: "On implicit functions" (preliminary communication).

In the absence of the authors, the papers of Dr. Sisam and Professor Carmichael were read by title. Also, owing to the length of the programme, the second paper of Professor Bolza and the paper of Professor Hedrick were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles listed above.

1. In this paper Dr. Neikirk gives a construction of the algebraic number systems of the form

$$\omega = \sum_{r=1}^{r=n} C_r j^{n-r},$$

where j is a root of the irreducible equation $f(x) = 0$ and the C's are integral. This was accomplished on the ordinary form of complex plane. The results were then applied to finite fields, such as the Galois field, by distinguishing two kinds of points; *i. e.*, reduced points and congruent points.

2. Mr. Schweitzer observes that his n-dimensional geometric K_n system* ($n \geqq 3$) forms a convenient descriptive basis for the consideration of the fundamental principles of the Ausdehnungslehre. In fact, the statement $\alpha_1\alpha_2 \cdots \alpha_{n+1}\, K_n \beta_1\beta_2 \cdots \beta_{n+1}$ may be identified with the statement

$$s(\alpha_1\alpha_2 \cdots \alpha_{n+1}) = t(\beta_1\beta_2 \cdots \beta_{n+1}),$$

where s and t denote real numbers such that $st > 0$ and the indicated $(n+1)$-ads denote elementary magnitudes of the $(n+1)$st grade.

3. Professor Davis calls the point (x', y') a black point. It is joined by a red line to the point $(x' + x'', y' + y'')$ called a blue point. Thus x' and y' are the real parts of the coordinates, while x'' and y'' are the imaginary parts of the coordinates of the point $(x' + ix'', y' + iy'')$.

If there be a linear relation between $x' + ix''$ and $y' + iy'$ then there is in general a (1, 1) correspondence set up between black points and blue points in the plane. But if the blue line joining two blue points be parallel to the black line joining the corresponding black points, then all the points satisfying the linear relation and no others are satisfied by the elements repre-

* Cf. BULLETIN, Nov., 1906.

sented by joining any point whatsoever of the black to any point whatsoever of the parallel blue line.

If the blue line coincide with the black line there results an ordinary real line.

4. The first paper by Professor Dickson gives a general investigation of the linear homogeneous groups on m variables with coefficients in a field F which are simply isomorphic with the symmetric group on q letters. In particular, it is shown that $m \geqq q-1$, unless F has a modulus dividing q. In the latter case there are representations on $q-2$ variables but not on fewer.

5. The paper of Professor Van Vleck, after briefly discussing a simple example of a continuous curve without a tangent, passed to the consideration of an example, simpler than any given hitherto, of a continuous curve which has an infinite number of oscillations in every part, however small, and yet possesses an ordinary tangent. This was followed by the following theorem: If in the interval (a, b) and in every subinterval the continuous curve $y = \phi(x)$ is of unlimited variation, there is a set of points of the second category, everywhere dense upon the curve, at which $\phi(x)$ has no derivative. This theorem may be considered as the complement of one previously obtained by Lebesgue for a continuous curve of limited variation.

6. Dr. Young called attention to the fact that certain of the known isothermic surfaces have isothermal asymptotic lines and undertook in his paper to show how all such surfaces may be obtained. He showed that the simplest treatment of the problem was from the standpoint of the spherical representation of the lines of curvature. For the problem reduces to that of finding all isothermic surfaces for which the spherical images of the lines of curvature are isotherms. Dr. Young obtained both the singular and general solution of an equation which expressed the condition of integrability of the Codazzi equations, and in this way reduced the problem to that of determining those forms of the arbitrary functions, thus obtained, which will satisfy the Gauss equation for the sphere. The singular solution leads to the quadrics, certain cyclides, certain surfaces associated with these, which Dr. Young has discussed in a previous paper, and finally to a number of new surfaces whose cartesian coordinates he has obtained. The latter furnish a clew

to two new sets of isothermic surfaces of which the surfaces found are particular cases. These he will report upon later.

The most general case arising from the general and singular solution combined Dr. Young has not completed, but he has carried the work far enough to indicate final success.

7. In his first paper Mr. Ingold exhibits a theory of vectors in a euclidean n-space S_n, as a special chapter of Professor Maschke's symbolic differental invariant theory. The derivatives $f_1, \cdots, f_n$ of the symbolic function f are identified with a set of n mutually orthogonal unit vectors of S_n; equivalent reference systems are introduced and symbolic expressions are obtained for λ-dimensional forms in S_n, inner and outer products, product determinant expansions, and regressive products. Finally the symbols of a λ-space A_λ contained in S_n are identified with a system of vectors tangent to the parameter curves of A_λ.

8. Lejeune Dirichlet has given expressions for the number of classes of forms of negative determinants $D = -P$ or $-2P$ in terms of the sum of the Jacobi symbols S/P for the octants and twelfths of P, and Dedekind has shown in his notes on Gauss's Works (Untersuchungen über höhere Arithmetik, pages 673–7, 693–5) how from these formulas Gauss's expressions for the distribution of the quadratic residues in the octants and twelfths of P may be obtained. In *Crelle's Journal*, volume 127 (1904), pages 1–19, Karpinski has extended the method of Dedekind to find the distribution of the quadratic residues in the 10ths and 24ths of P.

In the present paper, Mr. Escott has generalized the result of Dr. Karpinski for the division of P into $n$ths and has shown how these formulas will simplify in many cases the finding of the number of classes for a given determinant.

9. In his paper on "Sufficient condition for a minimum with respect to one-sided variations" (*Transactions*, volume 5, pp. 477–492) Professor Bliss has proved that under certain conditions a set of extremals tangent to a given curve form a field. This proof is indirect inasmuch as he proves the theorem first for the special case of curves representable in the form $y = f(x)$, and then reduces the general case of parameter representation to the former by a point transformation of the plane. The object of Professor Bolza's first note is to give a

direct proof of the theorem for the general case of parameter representation.

10. In a paper published in the last volume of the *Quarterly Journal of Mathematics* Mr. Bassett obtains the equation of a rational quintic curve having four cusps, from the fact that it is the dual of a rational quartic having one cusp. Dr. Field derived the equation and general form of the curve by inverting a tricuspid quartic which passes through the vertices and is tangent to one side of the triangle of reference.

11. Professor Dickson's second paper appeared in full in the May BULLETIN.

12. Dr. Sisam derived a formula for the number of tangents to an arbitrary algebraic curve such tha he tangents at two intersections of any one of them with the urve intersect on the curve. He then showed that when the given curve is a parabola or hyperbola of mth degree, the tangents at any pair of points lying on the same tangent meet on the curve, and conversely. Several interesting properties of the correspondences thus established between the points and tangents of the curve were also discussed.

13. Mr. Ingold's second paper contains generalizations of the ordinary vector differential geometry as given by Burali-Forti and Fehr. A system of vectors tangent to the parameter lines is taken as reference system on an $(n-1)$-dimensional surface: these are determined from the cartesian representation by the methods of the first paper.

14. The iterated generalized limits of a multiple sequence may be unequal. Moreover, the existence of iterated generalized limits does not necessitate the existence of the generalized limit of a multiple sequence, nor vice versa. Dr. Dodd extends his treatment of iterated limits (*Mathematische Annalen*, volume 61, Heft 1) to the generalized limits of Cesàro and Borel.

15. The fundamental theorem of Mr. Birkhoff's paper is concerned with the character of the solutions of a differential equation

$$(1) \qquad \frac{d^n z}{dx^n} + \rho a_{n-1}(x, \rho)\frac{d^{n-1}z}{dx^{n-1}} + \cdots + \rho^n a_0(x, \rho)z = 0$$

for large values of the complex parameter ρ, $a \leqq x \leqq b$. Here for $|\rho| \geqq R$

$$a_i(x, \rho) = \sum_{j=0}^{\infty} a_{ij}(x) \cdot \frac{1}{\rho} j.$$

The theorem holds on certain regions, $\theta \leqq \arg \rho \leqq \phi$, of the ρ-plane. With its aid is solved in large measure the problem to determine the distribution of those values of ρ for which a solution of (1) exists fulfilling n conditions

$$(2) \quad \sum_{j=0}^{n-1} \left\{ \alpha_{ij}(\rho) \frac{d^j z}{dx^j}(a, \rho) + \beta_{ij}(\rho) \frac{d^j z}{dx^j}(b, \rho) \right\} = 0 \quad (i = 1, 2, \cdots, n),$$

and to determine the character of these solutions. Here for $|\rho| \geqq R$

$$\alpha_{ij}(\rho) = \sum_{k=0}^{\infty} \alpha_{ijk} \cdot \frac{1}{\rho} k; \quad \beta_{ij}(\rho) = \sum_{k=0}^{\infty} \beta_{ijk} \cdot \frac{1}{\rho} k.$$

Using these facts, theorems analogous to those which Kneser has given in the case $n = 2$ are derived concerning the expansion of a function $f(x)$ in a series of solutions of a differential equation and conditions (like (1), (2))

$$(3) \quad \begin{aligned} & p_n(x) \frac{d^n z}{dx^n} + p_{n-1}(x) \frac{d^{n-1} z}{dx^{n-1}} + \cdots + p_0(x) z + \rho^n A(x) z = 0, \\ & \sum_{j=0}^{n-1} \left\{ \alpha_{ij} \frac{d^j z}{dx^j}(a, \rho) + \beta_{ij} \frac{d^j z}{dx^j}(b, \rho) \right\} = 0 \qquad (i = 1, 2, \cdots, n) \end{aligned}$$

in the real self- or opposite-adjoint case, $p_n(x) \neq 0$, $A(x) \neq 0$. It is also proved that this restriction to the real self- or opposite-adjoint case are to a great extent unessential.

16. The object of Professor Miller's paper is to develop theorems which will make the commutator a more useful instrument in the study of group properties. The results are analogous to those which relate to the groups of genus zero. The following theorems are proved: Any dihedral group whose order is divisible by 8 may be defined by the orders of two generators and the order of the commutator of these generators, and no other groups are known to have this property. There is an infinite system of groups such that each of them is generated by two operators whose commutator is of order 2 while the order of one of the generators is any given number which

exceeds 3 and the order of the other is any given even number greater than zero. If the orders of the two generators of a group are either 3, 4 or 3, 5 and if the order of their commutator is 2, these generators may be so selected as to give rise to an arbitrary group in an infinite system. If a group is generated by two operators of orders 2 and 3 respectively whose commutator is of order 3, its commutator subgroup is generated by two operators of order 3 whose product is also of this order. There are three and only three such groups which have an abelian commutator subgroup. The orders of these groups are 6, 18, and 54 respectively.

17. Professor Bolza's second note supplements Dr. Carathéodory's investigation concerning conjugate points for discontinuous solutions in the calculus of variations, by giving in explicit form the equation which connects the parameters of a pair of such conjugate points.

18. In a previous paper by Professor Laves on the same subject (presented at the Chicago meeting of April, 1906) it was shown that the partial differential equation for a body rotating about a fixed point can be integrated by determining the action of the system between two terminal positions. It is here shown how this proceeding rests on a theorem of Jacobi for problems depending upon two parameters. A family of orthogonal curves, one set of which are the extremals for the same value of the constant of energy, while the other are the transversals, covers the plane. The action of the system between two positions A_0 and A_1 is the difference between the values of the transversals passing through these same points. This is the form given to Jacobi's theorem by Sir William Thomson.

19. Professor Carmichael's first paper appeared in full in the March number of the *American Mathematical Monthly*.

20. In this table Professor Carmichael gives every possible value of Euler's function $\phi(m)$ up to 1000, and lists opposite each value all the numbers m which correspond to it. It is now more than 30 years since Cayley expressed a desire for the construction of this table. It will, it is believed, be found useful in various ways in the study of number theory.

21. In his third paper Professor Carmichael constructs the semicubical parabola by continuous motion, and applies the

locus to finding a cube having to a given cube any ratio which may be expressed by two straight lines.

22. The proposition, "If A, B, C are three points of one line and A', B', C' are three points of another line, then the intersections, $BC'/B'C$, $CA'/C'A$, $AB'/A'B$ lie on a line" (Salmon: Conics, § 268) puts in simple form Propositions 138, 139, Liber VII, Collectiones of Pappus. The figure arising is a special case of the Pascal hexagon which may be called the Pappus hexagon. Professor Grove's paper develops the complete Pappus hexagon and shows the duality of the figure in the following and other theorems with their duals:

I. Three points π_i on each of two lines D_1, D_2 meeting in the point δ_1 are cross-joined by nine lines which meet in 18 points 1–18, which are the vertices of two sets of three line triads each. The triangles of each set are *inter se* in triple perspective, having as axes of perspective the lines D_1 and D_2 each three times and three lines P_i on the point Σ_2 or Σ_1 respectively, and having as centers of perspective the point Σ_1 or Σ_2 three times each for the sets respectively and twelve other points all of which lie on a covariant line E_1 which is the polar of δ as to any of the six triangles. The 18 points lie by threes on six lines P_i, which are three and three on the points Σ_1 and Σ_2 above.

II. The points Σ_1 and Σ_2 and δ are the false vertices of the complete quadrangle of the Hessian pairs of the point triads π_i on D_1 and on D_2.

Following names given in the complete Pascal hexagon, he calls the three points π_1, π_3, π_5 on D_1, and π_2, π_4, π_6 on D_2 Pappus vertices; the three lines P_i on Σ_1 and Σ_2 each, he calls Pappus lines; the points Σ_1 Steiner points whose line is S_i, the lines D_1 Hessian diagonals. Taking the lines P_i from the above theorem and proceeding in the dual form, we get three points π_i' on each of the *same* two lines D_i etc., etc. The derived triads π_i' and the counter-triads of the original triads are in an involution whose double points are the polar pair of the intersection δ as to the original triads π_i. Six other theorems follow which deal with special forms.

This paper will appear in full in the May number of the *American Mathematical Monthly*.

23. In this paper Professor Hedrick shows that the ordinary statements of the fundamental theorem on the existence of a solution of the implicit equation $F(x, y) = 0$ and their demon-

strations are included in the statement and demonstration of the more general and simple theorem:

If $F(x, y) = 0$ is satisfied by $x = x_0$, $y = y_0$, and if $F(x, y)$ is a continuous function of x and a continuously increasing (or decreasing) function of y near (x_0, y_0), then there exists one single-valued solution $y = \phi(x)$ near (x_0, y_0) such that $y_0 = \phi(x_0)$ and $F(x, \phi(y)) = 0$; and that solution is continuous.

A discussion of the relation of this theorem to other forms was appended.

H. E. SLAUGHT,
Secretary of the Chicago Section.

ON A LIMIT OF THE ROOTS OF AN EQUATION THAT IS INDEPENDENT OF ALL BUT TWO OF THE COEFFICIENTS.

BY PROFESSOR R. E. ALLARDICE.

(Read before the San Francisco Section of the American Mathematical Society, February 23, 1907.)

AT the end of a paper by Dr. Landau,* it is shown that every equation of the form $ax^n + x + 1 = 0$ has a root whose modulus is not greater than 2, and that every equation of the form $ax^n + bx^m + x + 1 = 0$ has a root whose modulus is not greater than 8. The object of the present paper is to show that every equation of the form

$$ax^n + bx^m + cx^l + \cdots + a_1x + a_0 = 0$$

has a root whose modulus is not greater than

$$\left|\frac{a_0}{a_1}\right| \cdot \frac{n}{n-1} \cdot \frac{m}{m-1} \cdot \frac{l}{l-1} \cdots$$

whatever be the values of the coefficients $a, b, c, \cdots$; and that, for certain values of these coefficients, this limit is attained.

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*"Ueber den Picardschen Satz," *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, Jahrgang 51, 1906.

It is obvious that by reason of the substitution $a_1 x = a_0 y$, we may take $a_1 = a_0 = 1$.

The method of proof lies in showing that, by taking appropriate increments of the arbitrary coefficients, we may increase the modulus of any root of the proposed equation, unless the root in question is one of a set of equal roots, the number of which is greater by one than the number of the arbitrary coefficients. The sole difficulty lies in the consideration of roots of equal modulus, but different amplitudes.

1) Consider the equation

$$ax^n + x + 1 = 0.$$

Let $\rho\alpha$, $\rho\beta$ be two roots with common modulus ρ $(\alpha \neq \beta)$; then

$$a\rho^n\alpha^n + \rho\alpha + 1 = 0, \quad a\rho^n\beta^n + \rho\beta + 1 = 0.$$

$$\therefore \rho(\alpha^n\beta - \alpha\beta^n) + \alpha^n - \beta^n = 0$$

and

$$\rho(\alpha^{-n}\beta^{-1} - \alpha^{-1}\beta^{-n}) + \alpha^{-n} - \beta^{-n} = 0,$$

whence

$$\rho(\alpha\beta - 1)(\alpha^{n-1} - \beta^{n-1}) = 0.$$

If $\alpha^{n-1} - \beta^{n-1} = 0$, it follows also that $\alpha^n - \beta^n = 0$, which is impossible; hence $\alpha\beta - 1 = 0$, or the two roots are conjugate, and a must be real.

Now, putting $x = \rho(\cos\theta + i \sin\theta)$ in the given equation, equating to zero the real and imaginary parts, taking differentials, and eliminating $d\theta$, we may easily show that it is possible to increase the modulus of each of the roots $\rho e^{\theta i}$, $\rho e^{-\theta i}$ by giving a real increment to a. The coefficient of $d\rho$ in the relation between $d\rho$ and da cannot vanish.

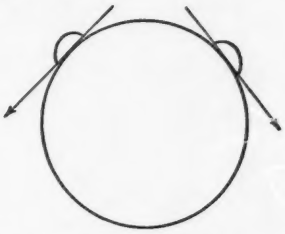
If, however, the proposed equation have two equal roots, an increment of a that will increase the modulus of one of these roots will diminish that of the other. Hence the proposed equation has always a root whose modulus is not greater than $n/(n-1)$, which is the value of the equal roots.

2) Consider now the equation

$$ax^n + bx^m + x + 1 = 0.$$

Let x_1, x_2 be two roots of equal modulus, neither of which is a multiple root; then

$$dx_1 = -\frac{x_1^n}{nax_1^{n-1} + mbx_1^{m-1} + 1}da - \frac{x_1^m}{nax_1^{n-1} + mbx_1^{m-1} + 1}db$$
$$= p_1 da + q_1 db, \quad \text{(say)}$$
$$dx_2 = p_2 da + q_2 db.$$



Now, representing x_1 and x_2 on a circle with radius equal to the common modulus, we see that $|x_1|$ and $|x_2|$ are increased if dx_1 and dx_2 lie in the spaces indicated *in* the figure.

We have

$$dx_1 = da\left(p_1 + q_1\frac{db}{da}\right), \quad dx_2 = da\left(p_2 + q_2\frac{db}{da}\right),$$

and hence, if db/da be taken arbitrarily, dx_1 and dx_2 may be made to rotate through four right angles, by varying the amplitude of da. It follows that, for some value of da, $|x_1|$ and $|x_2|$ will both be increased unless when dx_1 coincides with the tangent at x_1, dx_2 also coincides with the tangent at x_2 (in the directions indicated in the figure).

The conditions for these coincidences are

$$p_1 + q_1\frac{db}{da} = k_1 x_1 e^{-\frac{\pi}{2}i} e^{\alpha i}, \quad p_2 + q_2\frac{db}{da} = k_2 x_2 e^{\frac{\pi}{2}i} e^{\alpha i},$$

where k_1 and k_2 are real and have the same sign.

$$\therefore \frac{p_1 da + q_1 db}{p_2 da + q_2 db} = -\lambda\frac{x_1}{x_2} \quad (\lambda \text{ real and positive}).$$

Hence the fraction on the left must be independent of da and db, otherwise it would be possible to make it equal to any complex number.

$$\therefore p_1/q_1 = p_2/q_2, \quad \text{whence} \quad x_1^{n-m} = x_2^{n-m};$$
$$\therefore x_2 = \epsilon x_1, \quad \text{where} \quad \epsilon^{n-m} = 1.$$

It is easy now to deduce the values

$$x_1 = (1 - \epsilon^m)/(\epsilon^m - \epsilon), \quad x_2 = (1 - \epsilon^m)/(\epsilon^{m-1} - 1),$$

from which it follows that x_1 and x_2 are conjugate.

Denoting the conjugates of a and b by $\bar{a}$ and $\bar{b}$, we see that x_1 and x_2 are roots of the equations

$$ax^n + bx^m + x + 1 = 0, \quad \bar{a}x^n + \bar{b}x^m + x + 1 = 0$$

and hence of

$$(a\bar{b} - \bar{a}b)x^m + (a - \bar{a})x + (a - \bar{a}) = 0;$$

and the moduli of x_1 and x_2 may be increased together by the results obtained for the equation first considered. If a and b are both real, it may be shown as before, by putting $x = \rho(\cos\theta + i\sin\theta)$, differentiating and eliminating $d\theta$, that the moduli of x_1 and x_2 may both be increased by taking real increments of a and b.

Hence the modulus of any root of the given equation may be increased, unless it is a double root. But any double root of the given equation is a root of the equation

$$(n - m)bx^m + (n - 1)x + n = 0,$$

and may therefore have its modulus increased, unless it be a double root of this latter equation and therefore a triple root of the original equation. Thus the proposed equation always has a root whose modulus is not greater than $nm/(n - 1)(m - 1)$.

3) Consider now the equation

$$ax^n + bx^m + cx^l + x + 1 = 0.$$

As before, any two unequal roots of equal modulus may have their modulus increased unless

$$p_1 : q_1 : r_1 = p_2 : q_2 : r_2.$$

This leads to the equations

$$x_1^{n-m} = x_2^{n-m}, \quad x_1^{m-l} = x_2^{m-l},$$

which are impossible unless $n - m$ and $m - l$ have a common factor.

Let

$$n - m = k_1 r, \quad m - l = k_2 r$$

$$\therefore n = (k_1 + k_2) r + l, \quad m = k_2 r + l$$

and

$$x_2 = \epsilon x_1 \quad \text{where} \quad \epsilon^r = 1.$$

We may easily show that x_1 is determined by the equation

$$(\epsilon' - \epsilon) x_1 + \epsilon' - 1 = 0,$$

and that x_1 and x_2 must be conjugate; and the investigation may be completed as in the last case.

It is obvious that the above method may be continued so as to include equations containing any number of terms.

It may be stated in conclusion that the problem solved in the present paper is connected with the more difficult problem of determining a quantity ρ, a function of a_0 and a_1, such that there shall always be a root either of the equation $f(x) = a$ or of the equation $f(x) = b$, with modulus less than ρ, and that this latter problem is connected with the theorem of Picard, which is discussed in Dr. Landau's paper.

STANFORD UNIVERSITY,
February, 1907.

ON THE DISTANCE FROM A POINT TO A SURFACE.

BY PROFESSOR PAUL SAUREL.

(Read before the American Mathematical Society, April 27, 1907.)

IT is well known that in order that the distance from a given point to a given surface be a maximum or a minimum it is necessary that this distance be measured on a normal to the surface. But, so far as I know, the various possible cases have not been enumerated. This is done in the following theorem:

If P be an elliptic point of a surface, and if C_1 be the nearer and C_2 the more remote of the principal centers of curvature, the distance from a given point N of the normal to P will be a minimum if N and P lie on the same side of C_1, a maximum if N and P lie on opposite sides of C_2, and neither a minimum nor a maximum if N coincide with C_1 or C_2, or lie between them.

If P be a hyperbolic point of a surface, and if C_1 and C_2 be the principal centers of curvature, the distance from a given point N of the normal to P will be a minimum if N lie between C_1 and C_2, and neither a minimum nor a maximum if N coincide with C_1 or C_2, or lie without the interval C_1C_2.

If P be a parabolic point of a surface, and if C_1 be the principal center of curvature, the distance from a given point N of the normal to P will be a minimum if N and P lie on the same side of C_1, and neither a minimum nor a maximum if N coincide with C_1 or if N and P lie on opposite sides of C_1.

The demonstration rests upon the following theorem in plane geometry: In order that the distance from a given point to a given curve be a maximum or a minimum it is necessary that this distance be measured on a normal to the curve. The distance is a minimum if the given point N and the foot of the normal P lie on the same side of the center of curvature C, a maximum if they lie on opposite sides, and neither a minimum nor a maximum if N coincide with C.

We shall assume that this theorem is known. It may be well, however, to point out that it becomes almost intuitive if we observe that, in the immediate neighborhood of P, the curve lies outside of any tangent circle whose center lies on the same side of C as P, that it lies inside of any tangent circle whose center lies on the opposite side of C, and that it crosses the tangent circle whose center is at C.

The theorem in solid geometry follows at once from this theorem in plane geometry if we remember that for an elliptic point the centers of curvature of the normal sections lie within the interval C_1C_2, that for a hyperbolic point they lie without this interval, and that for a parabolic point they lie on that part of the normal which does not cut the surface.

It is worth noticing that the third part of the theorem can be considered as a limiting form of either the first or the second part.

NEW YORK,
April 13, 1907.

THE CALCULUS IN OUR COLLEGES AND TECHNICAL SCHOOLS.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY APRIL 27, 1907.

BY PROFESSOR WM. F. OSGOOD.

THE history of the race is frequently suggestive, in educational matters, of a wise course for the training of the individual. If we turn to the problems to which the calculus owes its origin, we find that not merely, not even primarily, geometry, but every other branch of mathematical physics — astronomy, mechanics, hydrodynamics, elasticity, gravitation, and later electricity and magnetism — in its fundamental concepts and basal laws contributed to its development and that the new science became the direct product of these influences.

Not until recent times have the analytic methods of the calculus been securely established. But at last the method of rates and that of infinitesimal constants have given place to the method of limits as being the only method known to us on which the calculus can be satisfactorily founded, satisfactorily, not merely from a logical point of view; it is important that the methods be in close touch with the physical concepts with which the calculus deals.

What is the bearing of these facts on instruction in the calculus in our colleges and technical schools? I wish to deal with some of the phases of this question.

§ 1. *Object of the Study of the Calculus.*

Let us first make clear to ourselves what some of the chief aims of the study of the calculus are. A view that has been widely accepted, to judge from the text-books that have had the largest sale, is that the first course should deal almost solely with the formal side. Just as the school boy learns to manipulate algebraic expressions with skill and accuracy, so the student shall first be drilled in differentiation and integration, the applications being chiefly to geometry and consisting in computing by formulas deduced in the text-book or class-room.

Recently the pendulum has swung to the other extreme. The reformists have discovered that the engineer is well off if he can plot some simple curves from the tables and differentiate and integrate x^2 and $\sin x$, and we are told that the race has been degenerating under the old regime. Let us not lose perspective and let us not for a moment fail to recognize the fact that, whatever changes it may be desirable to make in the suggestive instruction of the course, the process by which the youth actually acquires the ideas of the calculus is to a large extent and essentially through formal work of substantial character. In order to attain this end, however, the formal work must appear to him as having for its direct object the power to solve some of the real problems of pure and applied mathematics, and these problems must always be kept before his eye.

Setting aside such secondary aims as the ability to recognize the derivatives and integrals that enter in the text-books of applied mathematics, let us ask ourselves what enduring service the study of the calculus can render to the man whose life work calls for a qualitative, and, perhaps, to some extent quantitative appreciation of physical phenomena. I believe one of the most valuable contributions of the calculus toward this end to lie in the training by which the student ultimately reaches the point of being able to discern and formulate those concepts and relations of physics that find their natural expression in definite integrals and differential equations. The road to the qualitative is through the quantitative. To be conscious that the flight of a golf ball is governed by Newton's laws of motion and to discern what the forces are that act and how they bring about the slice, is to have taken an important step toward dealing intelligently with dynamical questions in the construction of machinery or of a steamship. But the problem of the golf ball should not come till many, many simple quantitative problems in rectilinear and circular motion have been worked out by the student.

§ 2. *Differentials, Ancient and Modern.*

The original conception of the infinitesimal was that it is a very small, an "infinitely small," *constant* quantity, but not quite the absolute zero, a sort of *little zero* that was insignificant when compared with any of the quantities of every day life except zero, but that two of these differentials could be compared with each other in their microcosmos just as the quantities of

every day life are comparable among themselves. Now this conception had its origin in a partial discernment of the truth, but, like the asymptotic series of astronomy, which converge only to a certain point and then diverge, it must not be pressed to the limit; for then the truth ceases and only chaos remains. Thus mathematicians have necessarily discarded the differentials of Leibniz as the elements out of which the calculus can be built up, and some are more than doubtful about the advisability of retaining them in any form.

We sometimes hear it said that hardly a theorem in our textbooks on the calculus is true as stated there. This is only a partial truth. Most of the theorems in the calculus are true when restricted as the authors intended they should be or as modern views readily suggest. And so here, in the case of the differentials, it behooves us to extract that which is true in the crude conception of former times and to shape a modern conception that is rigorous according to our present light.

There is a cogent reason for this course. The differentials are as firmly seated as ever they were in the methods of the physicists and engineers. We cannot banish them there, even if it were desirable to do so. Leave them out of the course in calculus and the student will meet them there with all the contradictions of the archaic infinitesimals, the little zeros of former times; and if he is capable of thinking clearly, he will see these contradictions and be helpless to reconcile them.

But beside this material reason there are two good scientific reasons for keeping the differentials and the infinitesimals. The former are useful in the formal work of differentiation and integration, the latter in the formulation of physical problems.

§ 3. *The Modern Conception of Infinitesimals.*

We have considered the archaic view of infinitesimals as small constants — a view still represented in the language, as when, for example, the President says: "The United States, owing to its peculiar position, has a regular army so small as to be *infinitesimal* when compared to that of any other first-class power." The modern scientific conception is that an infinitesimal is a *variable* which it is generally useful to consider only for values numerically small and which, when the formulation of the problem has progressed to a certain stage, is then allowed to approach zero as its limit.

Two infinitesimals, α and β, are said to be of the same *order* if their ratio, β/α, approaches a finite limit $K \neq 0$. If $K = 0$, β is said to be of higher order than α; and if $K = \infty$, then β is of lower order. If

$$\lim \frac{\beta}{\alpha} = 1, \quad \text{then} \quad \frac{\beta}{\alpha} = 1 + \zeta,$$

where ζ is infinitesimal, and hence α differs from β by an infinitesimal $\zeta\alpha$ of higher order than α or β. Conversely, if α or β differ from each other by an infinitesimal of higher order than that of either, then the limit of their ratio is unity. That infinitesimal which is chosen as independent variable is called the *principal infinitesimal*. If this be α, then, since

$$\lim \frac{\beta}{\alpha} = K, \quad \frac{\beta}{\alpha} = K + \zeta,$$

and

$$\beta = K\alpha + \zeta\alpha.$$

Thus β differs from $K\alpha$ by an infinitesimal of higher order than α; $K\alpha$ is called the *principal part* of β.

Two theorems lie at the foundation of the application of the calculus to natural philosophy.

THEOREM 1. In the limit of the ratio of two infinitesimals, either infinitesimal may be replaced by one that differs from it by an infinitesimal of higher order:

$$\lim \frac{\beta}{\alpha} = \lim \frac{\beta'}{\alpha'} \quad \text{if} \quad \lim \frac{\beta'}{\beta} = 1, \quad \lim \frac{\alpha'}{\alpha} = 1.$$

THEOREM 2. *Duhamel's Theorem.* If a sum of infinitesimals, all of like sign, approaches a limit, then each of the infinitesimals may be replaced by one that differs from it by an infinitesimal of higher order:

$$\lim_{n=\infty} (\alpha_1 + \alpha_2 + \cdots + \alpha_n) = \lim_{n=\infty} (\beta_1 + \beta_2 + \cdots + \beta_n),$$

if

$$\lim_{n=\infty} \frac{\beta_i}{\alpha_i} = 1.$$

§ 4. *The Use of Infinitesimals in Formulating Physical Problems.*

The importance of infinitesimals in the course in calculus which we are here considering lies in their use in formulating physical problems. A typical example or two will serve to make my meaning clear.

Let it be required to find the tension in a rope wound round a rough post and just on the point of slipping.

Consider a small arc PP' of the rope. We may regard it as a rigid body, since, if it became stiff, the forces would not necessarily thereby be disturbed. The forces that act on this

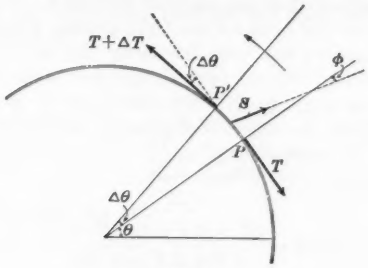


FIG. 1.

piece of the rope are: (1) the tension T along the tangent at P; (2) the tension $T+\Delta T$ along the tangent at P'; and (3) a resultant reaction S of the cylinder; — the weight of the rope is unimportant in comparison with the forces considered. Resolving forces along the tangent and the normal at P, we get *

$$\begin{aligned}(T+\Delta T)\cos\Delta\theta &= S\sin\phi + T,\\ (T+\Delta T)\sin\Delta\theta &= S\cos\phi.\end{aligned} \qquad (1)$$

Now if the rope is just on the point of slipping in the direction of the increasing angle θ, friction will act in the direction of the decreasing angle, and thus S will be inclined as shown in Fig. 1. What hypothesis shall we make about the direction of S? In other words, what is the physical fact? It is

* If the class is not familiar with the fact that, when a system of forces, acting on a rigid body, keep it at rest, they must be such a system as would be in equilibrium if they all acted at a single point, — then this theorem must, of course, be stated.

easy to see if we think of a heavy flexible string AB laid out straight on a rough horizontal table and pulled at one end A

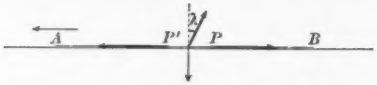


FIG. 2.

till it is just on the point of slipping. The total reaction S of the table on a segment PP' here obviously makes an angle ϕ with the normal equal to the angle of friction λ.

In the case of the cylinder we have similar relations. The rope presses against the cylinder along the arc PP', which, being short, is but slightly curved, and hence the direction of S will make an angle ϕ with the normal at P which differs only slightly from the angle of friction λ, and approaches λ as its limit when P' approaches P.

From equations (1) it follows that

$$\tan\phi = \frac{(T+\Delta T)\cos\Delta\theta - T}{(T+\Delta T)\sin\Delta\theta}$$

$$= \frac{\cos\Delta\theta}{T+\Delta T}\frac{\Delta T}{\sin\Delta\theta} - \frac{T}{T+\Delta T}\frac{1-\cos\Delta\theta}{\sin\Delta\theta}.$$

We are now ready to allow P' to approach P as its limit:

$$\lim_{PP'=0}\tan\phi =$$

$$\left(\lim_{PP'=0}\frac{\cos\Delta\theta}{T+\Delta T}\right)\left(\lim_{PP'=0}\frac{\Delta T}{\sin\Delta\theta}\right) - \left(\lim_{PP'=0}\frac{T}{T+\Delta T}\right)\left(\lim_{PP'=0}\frac{1-\cos\Delta\theta}{\sin\Delta\theta}\right).$$

The limit of $\tan\phi$ is μ, the coefficient of friction. The limit of the first factor in each product is obvious, namely $1/T$ and 1 respectively; for the tension increases gradually as we go along the rope, and so ΔT approaches 0 when P' approaches P.* In

* From equations (1) it can be proven that ΔT approaches 0 simultaneously with $\Delta\theta$; but the important view for the student at this stage is that the function T changes continuously because it represents a physical quantity that, from its very nature, varies continuously.

the limit of the second factor of the first term on the right, $\sin \Delta\theta$ can, by Theorem 1 of § 3, be replaced by $\Delta\theta$ and hence

$$\lim_{\Delta\theta=0} \frac{\Delta T}{\sin \Delta\theta} = \lim_{\Delta\theta=0} \frac{\Delta T}{\Delta\theta} = D_\theta T.$$

The same replacement can be made in the last limit on the right; and since

$$\lim_{\alpha=0} \frac{1 - \cos \alpha}{\alpha} = 0,$$

the limit of the whole last term on the right is 0. Our final result is, therefore,

$$\mu = \frac{D_\theta T}{T},$$

or

$$\frac{dT}{T} = \mu d\theta.$$

Hence, integrating, we get

$$\log T = \mu\theta + C,$$

and thus finally

$$T = T_0 e^{\mu\theta}.$$

In the foregoing solution, which I have given in all detail, it will be observed that, in the formulation of the problem, the physical law is clearly stated: ϕ is approximately equal to λ when P' is near P, and approaches λ as its limit when P' approaches P. A difficulty which the student frequently meets in this border ground between mathematics and physics is that of finding out what the physical fact is. The physicist often shrinks from writing down a formula for fear that the student will not understand what it means, quite losing sight of the fact that it is just at this point that the student can most effectively be aided in forming clean cut physical conceptions, the natural expression of which is through mathematical symbols.

The physical law once granted, equations (1) are absolutely true, whether the arc PP' subtends an angle of 10° or .000001″. Moreover, in taking the limits, nothing is "neglected." And the limits lead us directly to *derivatives, not to differentials.* The

differentials enter only subsequently through the derivatives. If the differentials are introduced at the outset, θ being taken as the independent variable, the force at P', namely $T + \Delta T$, is not $T + dT$; it differs from the latter quantity by an infinitesimal of higher order. And the equations (1), written with the Δ's replaced by the d's and ϕ by λ, are not true till a correction in the form of an infinitesimal of higher order has been added to one side of each equation. Thus the use of differentials in formulating the problem, if carried out rigorously, only detracts from the simplicity of the equations and at best replaces the quantities that naturally enter by artificial ones not quite equal to them. In the formulation of physical problems leading to definite integrals—areas and volumes, centres of gravity, moments of inertia, fluid pressures, attraction of gravitation, work and energy, kinetic energy, etc.—the situation is similar. First, the quantity to be computed is divided into n *infinitesimal* parts, *i. e.*, into n definite pieces, the magnitude of each of which is expressed accurately by an analytic symbol, and the quantity in question is precisely the sum of these pieces. Hence it is also equal to the limit of their sum. Then the ith part is approximated to by means of an infinitesimal which, in the case of a simple integral, is of the form $f(x_i)\Delta x_i$ and which differs from the ith infinitesimal in the above sum by an infinitesimal of higher order. By virtue of Duhamel's theorem the limit of the second sum,

$$\lim_{n=\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i,$$

is equal to that of the first sum, and thus the quantity to be computed is formulated as the definite integral

$$\int_a^b f(x)dx.$$

Here, again, nothing is "neglected." There is a clean cut formulation of the quantity in question as a sum, hence as the limit of that sum, and then follows the evaluation of this limit by means of a definite integral.

These are two of the central problems of the calculus, and an intelligent solution of them depends on a clear and adequate

understanding of what infinitesimals are and how to use them. No greater mistake can be made than to think that these problems belong in the applications of the calculus in the courses in physics and engineering. This view is a wrong one for two reasons. First, these problems are essential for understanding what the calculus is. Poincaré has said that it is nature which forced the continuum on mathematicians. If they had been left the prey of abstract logic, they would never have gotten beyond the theory of numbers and the postulates of geometry. Now, to understand the continuum, we must begin with continuous variation and the conception of the function. In no other way are these ideas brought home to our feeling, our intuition, as through the concepts and the simple laws of natural philosophy, beginning with the geometric problems of mensuration. To think that there is one calculus for the pure mathematician and another for the physicist, the engineer, the geometer, or the cultured layman, is to fail to appreciate that that which is most central in the calculus is its *quantitative* character, through which it measures and estimates the things of the world of our senses. And instruction in the calculus that does not point out — not merely at the beginning or at the end, but all through the course — this close contact with nature, has not done its duty by the student.

Thus we need these problems of natural philosophy in the course in calculus primarily for their own sake, alike for the training of the men who are to go on with mathematics and of those who are not. But there is another reason why we should treat these problems in the course in the calculus, and that is that the teachers in the courses in applied mathematics are frequently ignorant of modern methods in the calculus. Infinitesimals were given to them as small constants, zero, but yet not zero, that can be kept or thrown away at the pleasure of the instructor. They accepted the situation, and if their scientific consciences ever protested, they have long since been silenced. As I said at the beginning, there is a partial truth contained in this archaic view of the infinitesimal, and this is enough for some students. But our colleagues in applied mathematics make a huge mistake when they assume, as mathematicians once did, that their little zeros satisfy their best students. I remember hearing my father relate an amusing anecdote from his college days, which shows how abhorrent this juggling with infinites is to a healthy mind still able to

reject the poison. It was in the class in geometry and Professor P. had called on Mr. B. to prove the theorem by which the area of a circle is expressed in terms of the length of its circumference. Mr. B. had learned his lesson from the professor's text-book, in which it was set forth that a circle is a polygon of an infinite number of sides; and he recited the proof down to the conclusion, and there he stopped. "What does that prove?" asked the professor. "It proves an absurdity!" was the reply.

The science has advanced since then. We no longer have the infinite and the infinitesimal constants of former days in mathematics. If we are going to replace them in the courses in applied mathematics by the true infinites and infinitesimals, the variables, we must begin by teaching in the courses in mathematics what these variables are and how to use them.

§ 5. *The Place of the Differential.*

The course in calculus opens with finding the slopes of curves and the derivatives of the elementary functions, and makes application to maxima and minima and to some extent, too, to curve plotting; the more substantial problems in velocities coming a bit later. Here, the limit of the ratio, $\Delta y/\Delta x$, is the central idea, the key to the situation, and the attention of the student should not be embarrassed by any attempt to introduce infinitesimals at this first stage. $\Delta y/\Delta x$ is a function of the independent variable Δx and approaches, in all cases considered, a definite limit (or possibly, for certain values of x_0, becomes infinite) when Δx approaches 0. Its limit should be denoted by an expression that does not suggest a ratio. The expressions of this sort in common use are

$$D_x y, \quad y' \text{ or } y_x, \quad f'(x).$$

But the last three are too meagre a notation for the beginner, and so one naturally chooses $D_x y$.

Practical considerations make it necessary to introduce the differential at an early stage. The student meets the differential notation in his courses in applied mathematics and so it must be explained early in the course in the calculus. My experience, which I notice is confirmed in substance by that of recent writers, leads me to bring in the differential as soon as

the differentiation of the algebraic functions has been completed and the applications above mentioned taken up, but while there is still enough drill work in formal differentiation left so that the student will become thoroughly familiar with the method. Since we must teach differentials before the student is ready for them, we may at least get the advantage in technique which accrues from them in such differentiations as

$$d\sqrt{a^2 - x^2} = \tfrac{1}{2}\frac{1}{\sqrt{a^2 - x^2}}\, d(a^2 - x^2) = \text{etc.}$$

The differential is to be introduced as an infinitesimal, a variable, coordinate with the increments Δx, Δy, etc., and differing from its corresponding increment by an infinitesimal of higher order. dy is the principal part of Δy, Δx being taken as the principal infinitesimal. This is the first, but by no means the last word to be said in the course about infinitesimals, and the teacher must make this preliminary exposition brief, but incisive, a careful explanation of the usual geometric figure for illustrating $\Delta x = dx$, Δy, and dy being sufficient for the purpose.

In practice the important side of differentials is the formal side. They form one of the oldest examples of homogeneous variables and their chief value in the calculus consists in their application in formal differentiations and integrations. They appear to good advantage, too, in problems in velocities, for by their aid these problems can be reduced to solving some homogeneous equations in ds, dx, dy, $\cdots$, dt for ds in terms of dt.

§ 6. *The Service of Modern Analysis to the Calculus.*

A question that many of us have asked ourselves is: What service has modern progress in analysis rendered to the calculus of the undergraduate curriculum? At one time it looked as if recent advances in analysis were but to veil the conceptions of the calculus in a mist of ϵ's. But maturer thought disclosed that the formal ϵ, δ-proof is not needed till questions in double limits are reached, and the treatment of these questions belongs to a later stage. The fundamental idea of the limit is that a variable point, restricted according to the conditions of the problem under consideration, ultimately comes and remains within any arbitrarily assigned neighborhood of a certain fixed point, no matter how small this neighborhood may be chosen.

In the development of analysis during the nineteenth century this conception had to meet two demands. The first was that of its adaptability to the needs of the growing science; the second was the refinement which mathematics requires of the ideas which it accepts as fundamental. The fact that it stood both of these tests satisfactorily was an important factor in determining mathematicians to base the calculus on limits, and not on the infinitesimals of Leibniz or the rates of Newton.

A further service of modern analysis lies in its recognition of the elementary theorems about continuity as a secure foundation, a reliable method. If a function $f(x)$, continuous in the interval $a \leqq x \leqq b$, vanishes at the extremities of the interval and is positive at all interior points, or if, more generally, it has at some interior point a value larger than either $f(a)$ or $f(b)$, then it has a maximum value, and in fact a maximum maximorum, within the interval. Likewise, if $f(x)$ becomes positively infinite at each extremity of the above interval and is continuous at all intermediate points, it has a minimum value within the interval. These principles can be used with advantage in the problems in maxima and minima at the beginning of the course. It is important that the student should plot roughly the graphs of the functions he tests for maxima and minima, partly that he may avoid the error of mistaking a relative maximum or minimum at an interior point for a minimum minimorum at an extremity of the interval,* but more especially to give him practice in thinking of a function geometrically as a curve and in reasoning about its properties from inspection of the curve. Nearly all the cases that arise in practice, the "clothed problems," can be dealt with rigorously in this manner without the use of the second derivative, the reasoning being (1) that there is *at least* one maximum or minimum within the interval; (2) that the first derivative of the function (since it exists at each point within the interval) must vanish there; and then it turns out (3) that the latter equation has *only one* root in the interval.

Thus the second derivative is not usually needed for the problems of maxima and minima that arise in practice. It is, however, of value to have an analytic test for relative maxima

* Professor Huntington has given a neat example of this case. A bather standing on the bank of a circular pond, wishes to reach the point directly opposite in the shortest possible time. If he can swim at the rate of u miles an hour and run at the rate of v miles an hour, what point should he swim toward?

and minima. Now the most instructive form in which this test can be presented at this early stage in the course is as a problem in curve plotting, and this leads me to say a few words about the right and the wrong way to plot curves.

We need a few graphs accurately plotted because of the importance of the functions they represent; namely, the graphs of $\sin x$, $\cos x$, $\tan x$, $\log x$ and the inverse functions, and x^n, $n > 0$, $x \geqq 0$. These the student can plot directly from the tables (but he won't do it correctly unless the instructor has good staying powers). What we usually want to know about the graph of a function in the calculus is its character, and even when we need quantitative accuracy, a knowledge of the character of the curve at the outset is a help in plotting. Now the calculus affords a powerful aid for determining the character of a function. If $y = f(x)$ and $D_x y$ is positive, y is increasing with x; if negative, decreasing. Furthermore, if $D_x \tan \tau = D_x^2 y$ is positive, the slope of the curve is increasing with x and so the curve is *concave upward;* if $D_x^2 y$ is negative, the curve is concave downward. In addition to these principles we have the elementary theorems of modern analysis about continuous functions.

That my meaning may be perfectly clear at this point, consider the familiar example:

$$y = x^3 + px + q.$$

Here

$$D_x y = 3x^2 + p, \quad D_x^2 y = 6x.$$

From the last equation we see that for all negative values of x the curve is concave downward, for positive values concave upward. If $q = 0$, the curve is symmetric with respect to the origin; otherwise with respect to the point $x = 0$, $y = q$. Finally, since

$$y \,|_{x=-\infty} = -\infty,$$

we have the character of the curve in its entirety without computing the coordinates of a single point except those of the point of inflection. The slope at that point is p and thus if $p \geqq 0$, the cubic equation

$$x^3 + px + q = 0$$

has but one real root. If $p < 0$, the coordinates of the two maxima and minima tell the story. Other examples of this kind can serve as exercises and thus the graphs of polynomials and rational functions, *if treated by the methods of the calculus above mentioned*, can be used to advantage for training in the calculus.

It is a mistake, however, to dwell very long on these exercises near the beginning of the course. The student learns to apply the above principles by getting examples of their use when he is not looking for them. In one clothed problem or another he will come upon such an equation as

$$\theta - \tan\theta = 0.$$

One root is $\theta = 0$. Is there any further root in the interval from $\theta = 0$ to $\theta = \pi/2$? Let him work out this problem without any suggestions, or only with meagre ones. And when he has finally seen the point, let him find out how many roots the equation has in the interval from $\theta = \pi/2$ to $\theta = 3\pi/2$. Other equations of similar type are:

$$a\theta = \tan\theta \qquad (a \neq 1),$$

$$\theta(1 + \cos\theta) - 2\sin\theta = 0,$$

$$(1 + b^2)\tan^{-1} b = b.$$

And when the orthogonal families of confocal ellipses and hyperbolas are taken up, let him show analytically that the equation in λ:

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1,$$

has one and only one root in each of the intervals $-a^2 < \lambda < -b^2$ and $-b^2 < \lambda < +\infty$.

A simple problem in maxima and minima that calls for curve tracing as here set forth is the following: A gutter whose cross-section is an arc of a circle is to be made by bending into shape a strip of copper. If the width of the strip is a, find the radius of the cross-section when the carrying capacity of the gutter is a maximum.

In the second course in the calculus the Γ-function is introduced by the integral

$$\Gamma(\alpha) = \int_0 x^{\alpha-1}e^{-x}dx \qquad (\alpha > 0),$$

and its derivatives are given by differentiating under the sign of integration. Assuming this process to be justifiable, let the class determine the character of the graph of the function.

Here, then, is another great principle of the calculus. It is not quickly learned. To inculcate it into the student, it is not necessary, however, nor even desirable, to take any large amount of time from other things. What is required is vigilance on the part of the teacher. The psychological moment will not come with all students, or even many, in a given hour. It is necessary to give the class successive opportunities, spread out over the whole course, so that the environment may frequently be favorable for the reception of the desired suggestion. But when a student has once come to incorporate this principle into his mode of thought, he has received from the calculus an aid to that of which I spoke at the beginning — the power to grasp, through the quantitative, the qualitative side of the physical world about us.

§ 7. *Analytic Mechanics.*

A further subject of importance for the training of the undergraduate in mathematics is that of mechanics. This subject is treated most satisfactorily, I believe, in a collateral course in charge of the department of mathematics, provided that the instructor in this course has clear insight into the elementary facts of mechanics and their exemplification in every day life, and is interested primarily in teaching *mechanics*, not analytic geometry and the calculus. It is desirable to begin with the statics of particles and rigid bodies, treated both analytically and graphically. In the statics of a particle the force polygon, drawn to scale, serves as a check on the analytic solution. Half a dozen lectures on the graphical statics of roof and bridge trusses, including the graphical determination of centres of gravity, prove highly interesting to a college class. Thus the student has been clarifying his ideas of force before the dynamics of moving bodies is begun, and in the meantime he has learned in the course in the calculus how to differentiate and inte-

grate. He is now ready for Newton's second law of motion, a desirable foundation for dynamics for the beginner, since it makes him think about inertia and the way a body moves under the action of forces, as well as about the reaction of the parts of a machine on their constraints. The course naturally includes an extended treatment of centrifugal force, the principle of work and energy, and the dynamics of a rigid body in the case of motion in two dimensions. It is also important to set forth the conceptions and the fundamental theorems about the instantaneous center, the space and the body centrode, and the law of cam motion.

From the time dynamics is begun, — about one third of the course should be given to statics, — the calculus is of vital importance. On the other hand the problems, several of which should be worked independently by the student in connection with each lecture, ought not, as a rule, to involve serious difficulties of an analytic nature. There ought to be many easy and some hard exercises; but the difficulty of the latter should be due to the incompleteness of the students mechanical intuition, not to complicated analytic or geometric considerations.*

The course in the calculus ought to contain a chapter in mechanics, in which rectilinear motion under forces that are functions of the space or of the velocity are treated and which thus supplements the course in mechanics without duplicating it. If, however, such a course in mechanics as the above is not given, this chapter must be lengthened; but it cannot in any case replace the course in mechanics. I have for many years given both courses, and I have found the course in mechanics to afford most valuable training in the appreciation of some of the leading principles of the calculus; and this in spite — or perhaps rather because — of the fact that I reduce the analytic difficulties of the course to a minimum.

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* In technical schools the course in graphical statics, if prescribed for all who take this course, makes it possible to abridge slightly the treatment of statics. But the course we are here considering is otherwise independent of that course and the course on the strength of materials.

It is, of course, desirable that a second course in mechanics be given. But inasmuch as a first course of the elementary character here outlined is frequently lacking and is for most students the more instructive of the two, — it is a far more difficult course to give effectively than the advanced course is, — I am concerned here with emphasizing this gap in mathematical instruction.

§ 8. *Advanced Calculus.*

The first course in the calculus, even though it be a five hour course preceded by differentiation in the freshman year, is not adequate for covering all the topics that belong in a proper treatment of the elementary part of the subject. Take, for example, partial differentiation. The easier applications to problems in the geometry of space can be treated in the first course. But the more difficult applications which arise in pure mathematics and in physics, and which are largely formal, clearly should find a place in the second course. Again, in multiple integrals, the conception of the multiple integral, its evaluation by means of iterated integrals, and its use in formulating physical problems belong early, and in technical schools the instruction should be so arranged that all students of engineering will get this part of the subject. But Green's theorem and its applications to exact differentials and to such problems as the deduction of Laplace's equation as the condition for a steady flow of heat or electricity claim an important place in the second course.

Then there are the hyperbolic functions, the elliptic integrals and functions, and the Γ- and B-functions, the latter being defined by definite integrals. When I say that the course should treat thoroughly the simplest tests for the convergence of improper definite integrals, you will all agree with me and some will wonder why I do not demand more, tests for continuity, etc. The reason is that the student is not ready to receive more at this stage. If those of you who give a second course in the calculus will inform yourselves about the student's real power in dealing with the simplest cases where he has to detect unaided whether the integral in question is a proper or an improper integral, you will find a degree of immaturity hard to realize. He will test, for example, the integral

$$\int_0^1 \frac{\log x}{1-x}\,dx$$

for convergence or divergence at the upper end of the interval of integration with the same care that he bestows on the lower end. "Have him plot the graph of the integrand," you say. Good advice; this is an aid. But ask the class to note the time they put on the exercises that are handed in, and take the lesson to heart. The fact is that the student is nowhere near

the point where he can even see at a glance in all the simpler cases whether the integrand becomes infinite or remains finite; much less is it easily apparent to him whether to expect convergence or divergence, and what test to apply. Now these are the very things that have to be taught him right here and it takes time. But it is time well spent; it is the teacher's opportunity to strengthen the student's power to deal with functions and at the same time to treat the formal side of the definition of functions by means of definite integrals, the only questions of convergence here considered being those of the mere convergence of the integrals that present themselves. How far beside the mark double limits and ϵ-proofs are at this stage is clear to every teacher who has perceived his opportunity at this point.

The value of a special course on elementary differential equations is generally recognized. But, wholly aside from this course, differential equations should permeate both courses in the calculus. Through them clothed problems of first importance for training in the calculus are introduced.

I have not even mentioned the subjects of the further elementary developments of infinite series, including Fourier's series, of mean value and probability, of the approximate solution of equations, and of a brief introduction to the theory of functions of a complex variable and the calculus of variations. And yet most of these subjects should be studied in an elementary way before the more difficult questions of modern analysis are begun.

This, then, is the elementary material that forms the subject matter of the calculus. To introduce the treatment of questions in double limits at this stage is both to deprive the student of the chance to acquire a broad knowledge of the calculus and to bungle his training in analysis.

§ 9. *Conclusion.*

It has not been my intention to mark out a course in the calculus, for the great body of the instruction at present given by any good teacher is sound and ought to be retained in substance. Before all, there is the necessity of faithful insistence on the part of the instructor on accurate formal work of reasonably substantial character. The technique of the calculus must be taught. Then there is the large body of easy problems in

the applications which depart so slightly from the typical problems of the text-books or the class-room that they are within the reach of all who make an honest effort to do the work of the course. Moreover, as regards the choice of material, the important improvement of introducing the integral as the limit of a sum early in the first course may now be fairly regarded as achieved. And yet, with all of this that is so good and sound, if you open one of our text-books on the calculus and ask: What *is* the calculus? What will abide after the formulas are forgotten? What is the soul and the spirit of this great science, as conceived by the man whose work in life does not lie within the field of mathematics? I can't help feeling that the answer does not ring clear: The calculus is the greatest aid we have to the appreciation of physical truth in the broadest sense of the word.

NOTES.

THE fourteenth summer meeting of the AMERICAN MATHEMATICAL SOCIETY will he held at Cornell University on Thursday–Friday, September 5–6. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary by August 24. Abstracts intended to be printed in the announcement of the meeting must be submitted by August 17.

THE April number (volume 8, number 2) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Dynamical trajectories: the motion of a particle in an arbitrary field of force," by E. KASNER; "A class of periodic orbits of an infinitesimal body subject to the attraction of n finite bodies," by W. R. LONGLEY; "A proof of some theorems on pointwise discontinuous functions," by E. B. VAN VLECK; "Invariants of binary forms under modular transformations," by L. E. DICKSON; "Projective differential geometry of curved surfaces (first memoir)," by E. J. WILCZYNSKI; "A method for constructing the fundamental region of a discontinuous group of transformations," by J. I. HUTCHINSON; "Oblique reflections and unimodular strains," by E. B. WILSON; "On the introduction of convergence factors into summable series and summable integrals," by C. N. MOORE.

THE pages of "New Publications" contained in each number of the BULLETIN will hereafter be reprinted and furnished to such institutions as may desire to use them for the purpose of ordering mathematical books. Enquiries should be addressed to the Committee of Publication.

ON the editorial staff of the *Annals of Mathematics* Professor J. K. WHITTEMORE will succeed Professor MAXIME BÔCHER, who is about to assume his duties as member of the Editorial Committee of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY.

AT the meeting of the London mathematical society held on April 11, the following papers were read: By H. F. BAKER, "A theorem in the theory of functions"; by H. BATEMAN, "An introduction to the metrical geometry of space of n dimensions," and "The values of the parameters for which a definite integral can be zero"; by H. HILTON, "A note on Perott's theorem"; by E. W. HOBSON, "Poisson's integral and its relations to the proof of Fourier's theorem."

THE supplementary announcement of the annual list of members of the Circolo matematico di Palermo issued in March shows that the society now has 461 active members, of whom 75 are Americans. This society was organized in the spring of 1884 by residents of Palermo; it at once became national and soon international in its scope. At present 288 of its members are foreigners. The regular sessions of the society are held on alternate Sunday afternoons. The first volume of the *Rendiconti* was issued in 1887; at least one volume has appeared each succeeding year, and during each of the years 1905 and 1906 two volumes were published. The parts appear bimonthly; they contain brief reports of the sessions and such memoirs read before the society as are appropriate. Beginning with January, 1907, the reports were transferred to the *Supplemento*, which also gives the titles of papers published in other mathematical journals. The *Annuario* contains the list of members and a complete index of all papers published in the *Rendiconti*. In the exchange list of the society are 105 periodicals, including all the important mathematical journals of the world. The society, together with the Accademia dei Lincei at Rome, will have charge of the arrangements for the

fourth international congress of mathematicians to be held at Rome next April. The present officers of the Circolo are: president, Professor M. L. ALBEGGIANI; secretaries, Professors G. DI SIMONE and E. P. GUERRA; director of the *Rendiconti*, Professor G. B. GUCCIA.

ON the occasion of the Abel centenary celebration it was decided to erect a memorial tablet to his honor, and funds were subscribed for the purpose. This plan has now been succeeded by another, for which larger means are needed. A plaster cast of a monument 40 feet high representing Abel supported by gigantic allegorical figures has been prepared, which is to be cast in bronze. The work has been done by Gustav Vigeland, perhaps the ablest sculptor in Norway. In order that this undertaking may assume an international character, mathematicians throughout the world are invited to contribute to it. The circular of invitation bears the names of a responsible committee. Contributions may be sent to Professor L. Sylow, University of Christiania, Norway.

THE following advanced courses in mathematics are offered during the academic year 1907–1908:

COLUMBIA UNIVERSITY. — By Professor F. N. COLE: Theory of groups, three hours; Theory of invariants, three hours. — By Professor JAMES MACLAY: Application of the calculus to the theory of surfaces and curves in space, three hours; Theory of functions of a complex variable, three hours. — By Professor C. J. KEYSER: Modern theories in geometry, three hours; General theory of assemblages, three hours. — By Professor H. B. MITCHELL: Vector analysis, two hours; Differential equations, two hours. — By Professor EDWARD KASNER: Differential equations and continuous groups, three hours; General introduction to higher mathematics, three hours. — By Dr. G. H. LING: Modern higher algebra, three hours. — By Professor M. I. PUPIN: Theory of the potential function, two hours; Hydrokinetics, two hours; Partial differential equations of physics, two hours; Special problems, two hours. — By Professor A. P. WILLS: Mechanics, two hours; Theory of elasticity, two hours; Electricity and magnetism, electro-magnetic theory of light, two hours; Thermodynamics, two hours.

CORNELL UNIVERSITY. — By Professor L. A. WAIT: Advanced analytic geometry, three hours. — By Professor J. M. MAHON: Theory of potential and spherical harmonics, three hours. — By Professor J. H. TANNER: Theory of equations, two hours; Teachers' course in algebra, two hours. — By Professor J. I. HUTCHINSON: Automorphic functions, three hours. — By Professor V. SNYDER: Algebraic curves, three hours. — By Professor W. B. FITE: Advanced calculus, three hours; Theory of functions of a real variable, two hours. — By Dr. F. R. SHARPE: Theory of electrons, three hours. — By Dr. W. B. CARVER: Projective geometry, three hours. — By Dr. A. RANUM: Differential equations, two hours. — By Dr. D. C. GILLESPIE: Advanced differential equations, two hours. — By Mr. C. F. CRAIG: Theory of probabilities and insurance, two hours. The Oliver mathematical club will meet weekly.

JOHNS HOPKINS UNIVERSITY. — By Professor F. MORLEY: Vector analysis, two hours; Higher geometry, two hours; Seminar, one hour; Classic authors, one hour. — By Dr. A. COHEN: Differential equations, two hours; Elementary theory of functions, two hours; Introduction to differential equations and vector analysis, two hours. — By Dr. A. B. COBLE: Cremona transformations, two hours (first half year); Theory of statistics, two hours (second half year).

UNIVERSITY OF PENNSYLVANIA. — By Professor E. S. CRAWLEY: Solid analytic geometry, two hours; Higher plane curves, three hours. — By Professor G. E. FISHER: Theory of functions of a complex variable, first half year, three hours; Elliptic functions, second half year, three hours. — By Professor I. J. SCHWATT: Definite integrals, three hours. — By Professor G. H. HALLETT: Lies theory of continuous groups, first half year, three hours; Galois theory of algebraic equations, second half year, three hours. — By Dr. F. H. SAFFORD: Curvilinear coördinates, three hours. — By Dr. O. E. GLENN: Higher algebraic equations, two hours.

AT the meeting of the National academy of sciences held in Washington April 16 and 17, Professor A. R. FORSYTH, of Cambridge University, and Professor D. HILBERT, of the University of Göttingen were elected to associate membership.

PROFESSOR J. TANNERY, of the University of Paris, has been elected a member of the Paris academy of sciences.

PROFESSOR A. LOEWY, of the University of Freiburg, has been commissioned to give regular courses in the mathematics of insurance.

PROFESSOR G. DARBOUX has been appointed associate member of the Belgian academy of sciences.

PROFESSOR PH. FURTWÄNGLER, of the agricultural academy at Poppelsdorf has been appointed professor of mathematics at the technical school of Aachen.

PROFESSOR S. GUNDELFINGER, of the technical school at Darmstadt, retired from active service April 1, on account of ill health.

DR. W. LUDWIG, of the technical school at Karlsruhe, has been appointed professor of descriptive geometry at the technical school of Braunschweig.

PROFESSOR E. VON WEBER, of the University of Munich, has been appointed associate professor of mathematics at the University of Würzburg.

PROFESSOR K. WEIGHARDT, of the technical school at Braunschweig, has been appointed professor of mathematics at the technical school of Hanover.

DR. P. KOEBE has been appointed docent in mathematics at the University of Göttingen.

AT McGill University, W. M. EDWARDS, lecturer in mathematics, has been made assistant professor of civil engineering.

PROFESSORS T. S. FISKE and D. E. SMITH, of Columbia University, will be abroad on leave of absence during the academic year 1907–08.

AT the University of Nebraska, Professor A. L. CANDY has been promoted to a full professorship of mathematics and Professor C. C. ENGBERG to a full professorship of applied mathematics.

DR. S. D. TOWNLEY, astronomer in charge of the International Latitude Observatory at Ukiah, California, and lec-

turer in astronomy in the University of California, has been appointed to an assistant professorship in the department of applied mathematics at Stanford University.

DR. G. D. BIRKHOFF has been appointed instructor in mathematics at the University of Wisconsin.

MR. W. M. CARRUTH, of Cornell University, has been appointed assistant professor of mathematics at Hamilton College.

AT the Massachusetts Institute of Technology the following appointments have been made: Dr. E. B. WILSON, of Yale University, associate professor of mathematics; Dr. H. B. PHILLIPS, of the University of Cincinnati, and Mr. N. J. LENNES, of the University of Chicago, instructors in mathematics. Dr. C. L. E. MOORE has been granted a leave of absence and will spend next year studying in Italy and in Germany.

PROFESSOR MANSFIELD MERRIMAN has retired from the professorship of civil engineering at Lehigh University after a record of twenty-eight years' service.

CHARLES H. HINTON, author of several popular works on the fourth dimension and of late an examiner in the Patent Office at Washington, died in that city on April 30, 1907, at the age of 63 years.

DR. A. FUHRMANN, emeritus professor of mathematics at the Dresden polytechnic school, well known for his work on mechanics, died April 30 at the age of 67 years.

PROFESSOR FERDINANDO ASCHIERI, of the University of Pavia, died April 14 at the age of 60 years.

THE death has recently been announced at the age of 91 years of Professor M. Steinschneider, of Berlin, one of the best authorities on the history of Hebrew mathematics. Professor Steinschneider published numerous works on Hebrew and Hebrew-Arabic literature, including catalogues of the Hebrew manuscripts in various large libraries.

CATALOGUES of second-hand mathematical works: Gustav Fock, Schlossgasse 7, Leipzig, catalogue No. 300, 3,255 titles in mathematics and natural science; K. F. Kohler's Antiquarium, Kurprinzstrasse 6, Leipzig, catalogue No. 569, periodicals and proceedings of societies, about 2,000 titles; Martin Boas, Karlstrasse 25, Berlin, N. W., catalogue No. 60, astronomy, mathematics and physics; H. Lüneburg, Karlstrasse 4, Munich, catalogue No. 73, about 1900 titles in mathematics, astronomy, and physics, including works from the library of the late Professor G. Bauer.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CAJORI (F.). See CANTOR (M.).

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. Herausgegeben von M. Cantor, unter Mitwirkung von V. Bobynin, A. von Braunmühl, F. Cajori, u. a. Vol. IV: Von 1759–1799. Lieferung 1: XIX. Abschnitt: S. Günther, Geschichte der Mathematik; XX. Abschnitt: F. Cajori, Arithmetik, Gleichungslehre, Zahlentheorie. Leipzig, Teubner, 1907. 8vo. 200 pp. M. 5.60

——. Vorlesungen über Geschichte der Mathematik. Band I: Von den ältesten Zeiten bis zum Jahr 1200 nach Christus. 3te Auflage. Leipzig, Teubner, 1907. 8vo. 6 + 941 pp., 1 plate. M. 24.00

DU PASQUIER (L. G.). Zahlentheorie der Tettarionen. (Diss.) Zürich, 1906. 8vo. 75 pp.

FLECHSIG. See LUCZAC.

FORSYTH (A. R.). Theory of differential equations. Part IV: Partial differential equations. 2 vols. Cambridge, 1907. 8vo. Cloth.

GÜNTHER (S.). See CANTOR (M.).

HALBWACHS (M.). Leibniz. Paris, Delaplane, 1906. 18mo. 124 pp. F. 0.90

LAISANT (C. A.). La mathématique. Philosophie. Enseignement. 2e édition, revue et corrigée. Paris, Gauthier-Villars, 1907. 8vo. 7 + 244 pp. Boards. F. 5.00

LARMOR (J.). See STOKES (G. G.).

MARC (L.). Aufgaben aus der höheren Mathematik, technischen Mechanik und darstellenden Geometrie, welche bei der Vorprüfung für Bau-, Maschinen-, Elektro-, Kultur- und Vermessungs-Ingenieure sowie Architekten an der k. technischen Hochschule zu München vom Jahr 1901 ab gestellt worden sind. München, Lachner, 1907. 8vo. 48 pp. M. 2.00

ROMANO (F.). Sopra una trasformazione doppia del terzo ordine nei punti dello spazio. Avola, Piazza, 1906. 8vo. 47 pp.

SISAM (C. H.). On septic scrolls having a rectilinear directrix. (Diss., Cornell University.) Baltimore, 1907. 4to. 52 pp. (*American Journal of Mathematics*, Vol. 29, pp. 48–100.)

SOMMER (J.). Vorlesungen über Zahlentheorie. Einführung in die Theorie der algebraischen Zahlkörper. Leipzig, Teubner, 1907. 8vo. 6 + 361 pp. Cloth. M. 6.00

STOKES (G. G.). Memoirs and scientific correspondence of Sir George Gabriel Stokes. Selected and arranged by J. Larmor. 2 vols. Cambridge, 1907. 8vo. 1004 pp. 4 plates. Cloth.

THAER (C.). Ueber Invarianten, die symmetrischen Eigenschaften eines Punktsystems entsprechen. Leipzig, Teubner, 1906. 8vo. 31 pp. M. 0.80

VOGT (H.). Eléments de mathématiques supérieures, à l'usage des physiciens, chimistes et ingénieurs et des élèves des Facultés des sciences. 4e édition, très augmentée et entièrement refondue. Paris, Vuibert, 1907. 8vo. 8 + 712 pp.

VOGT (W.). Korrelative Räume bei gegebener Punktkernfläche. (Diss.) Breslau, 1906. 8vo. 69 pp.

WHITE (M. B.). The asymptotic lines on the anchor ring. (*Annals of Mathematics*, April, 1907.) 4to. 14 pp.

WIESING (O.). Ueber eine zwei-zweideutige Verwandtschaft zwischen zwei Ebenen und ihr Analogon im Raume. (Diss.) Breslau, 1906. 8vo. 83 pp.

II. ELEMENTARY MATHEMATICS.

DURELL (F.). Key to plane and solid geometry. New York, Maynard, 1906. 12mo. 173 pp. Cloth. $1.00

GORSE (F.). A school algebra course. Cambridge, University Press, 1907. 12mo. 312 pp. Cloth. 3s.

HAMMER (E.). Handbuch der ebenen und sphärischen Trigonometrie. Zum Gebrauch beim Selbstunterricht und in Schulen, besonders als Vorbereitung auf Geodäsie und sphärische Astronomie. Stuttgart, Metzler, 1907. 3te, erweiterte Auflage. 8vo. 18 + 644 pp. M. 10 60

JACKSON (W. H.). Elementary solid geometry, including the mensuration of the simpler solids. London, Arnold, 1907. 12mo. 172 pp. 2s. 6d.

JACQUET (E.) et LACLEF (A.). Compléments d'arithmétique, d'algèbre, de géométrie, accompagnés de très nombreux problèmes donnés dans les examens, spécialement rédigés à l'usage des candidats aux écoles des arts et métiers. Paris, Nathan, 1907. 16mo. 388 pp.

LACLEF (A.). See JACQUET (E.).

MENGER (J.). Leitfaden der Geometrie für Gewerbeschulen. 4te Auflage. Wien, Hölder, 1907. 8vo. 4 + 77 pp. Boards. M. 0.96

MINET (A.) et PATIN (L.). Cours pratique d'arithmétique, de système métrique et de géométrie. Paris, Nathan.

MINET (A.) et PATIN (L.). Cours élémentaire (1re et 2e années). 1906. 16mo. 240 pp. F. 2.25

——. Cours moyen. 1907. 16mo. 288 pp. F. 1.25

MORF. See TZAUT (S.).

PATIN (L.). See MINET (A.).

SUPLEE (H. H.). Five figure logarithms. Philadelphia, Lippincott, 1906. 16mo. Leather. $2.00

TZAUT (S.) et MORF. Exercices et problèmes d'algèbre, et réponses. (Première série.) 4e édition. Paris, Gauthier-Villars, 1906. 16mo. F. 5.00

III. APPLIED MATHEMATICS.

ADAMS (C. L.). Notes on descriptive geometry. Part I. Prepared for use in Massachusetts Institute of Technology. Boston, Adams, 1907. 8vo. Cloth. $1.50

ALBRECHT (M. F.) und BIEROW (C. S.). Lehrbuch der Navigation und ihrer mathematischen Hilfswissenschaften. Für die k. preussischen Navigations-Schulen. 9te Auflage. Berlin, Decker, 1906. 8vo. 22 + 528 pp. M. 14.00

BARTLETT (G. M.). Numerical problems in descriptive geometry, for class and drawing room practice. Revised edition. Ann Arbor, Bartlett, 1907. 83 pp. Cloth. $0.50

BENISCHKE (G.). Die wissenschaftlichen Grundlagen der Elektrotechnik. 2te erweiterte Auflage von "Magnetismus und Elektrizität mit Rücksicht auf die Bedürfnisse der Praxis." Berlin, Springer, 1907. 8vo. 15 + 580 pp. M. 12.00

BIEROW (C. S.). See ALBRECHT (M. F.).

BROGGI (V.). Matematica attuariale: teoria statistica della mortalità; matematica delle assicurazioni sulla vita. Milano, Hoepli, 1906. 15 + 347 pp. L. 3.50

CHAMPION (E.). Principes généraux d'électricité théorique et pratique, à l'usage des élèves des écoles industrielles. Vol. III: Dynamos et moteurs à courant continu; alternateurs monophasés et polyphasés; moteurs synchrones; transformateurs. Lille, Danel, 1907. 4to. 152 pp.

FISCHER (O.). Kinematik organischer Gelenke. Braunschweig, Vieweg, 1907. 8vo. 12 + 261 pp. (Die Wissenschaft; Sammlung naturwissenschaftlicher und mathematischer Monographien, Heft 18.) M. 8.00

GENOVINO (G.). Influenza dell'attrazione del sole e della luna sulla direzione della verticale, sulla gravità e sulla marcia dei pendoli. Firenze, Landi, 1907. 8vo. 64 pp.

HELLER (A. H.). Stresses in structures and the accompanying deformations. Columbus, Green, 1907. 8vo. 324 pp. Cloth. $4.00

JÄGER (G.). Theoretische Physik. Band II: Licht und Wärme. Band III: Elektrizität und Magnetismus. 3te verbesserte Auflage. Leipzig, Göschen, 1907. 16mo. 302 pp. (Sammlung Göschen.) M. 1.60

JORDAN (W.). Handbuch der Vermessungskunde. Band III: Landes-Vermessung und Grundaufgaben der Erd-Messung. 5te erweiterte Auflage. Bearbeitet von C. Reinhertz mit Vorwort von E. Hammer. Stuttgart, Metzler, 1907. 8 + 678 pp. M. 15.00

LIAPOUNOFF (A.). Sur les figures d'équilibre peu différentes des ellipsoïdes d'une masse liquide homogène douée d'un mouvement de rotation. Partie I: Etude générale du problème. St. Petersbourg, 1907. 4to. 225 pp. F. 7.00

MALCOLM (C. W.). Elements of graphic statics. Champaign, Malcolm, 1907. 8vo. 3 + 127 pp. Cloth. $1.75

MATRICULATION mechanics papers. London, Clive, 1907. 12mo. 1s. 6d.

PIERS (F.). Piers' mathematics for the machine shop: a pocket book for the use of machinists and tool makers. Philadelphia, Piers, 1906. 16mo. 78 pp. $1.00

RICHARD (G.). Mécanique. A l'usage des ingénieurs, constructeurs-mécaniciens, industriels, chefs d'ateliers et contremaîtres. 29e édition. Paris, Dunod et Pinat, 1907. 16mo. 5 + 197 pp. Fr. 2.50

VALLEREY (J.). Géométrie et mécanique. 2e édition, revue et corrigée conformément aux programmes. Paris, Challamel, 1907. 8vo. 522 pp. F. 7.50